The Cuntz semigroup, a Riesz type interpolation property, comparison and the ideal property

Cornel Pasnicu

University of Texas at San Antonio

This is joint work with *Francesc Perera*.

In this talk we'll:

• define a *Riesz type interpolation property* for the *Cuntz semigroup* $W(A)$ and prove that it is satisfied in the case when A has the *ideal property*.

• find *characterizations* of the *ideal property* in terms of the *Cuntz semigroup* (and several more in the stable, *purely infinite* case).

• define "*comparison*" and prove *comparison results* for classes of C^*-alg. A with the *ideal property* (including situations when A is an AH alg. with the *ideal property*).

Elliott's Program:
Classify sep., nuclear C^*-alg. by discrete invariants including K-theory.

Counterexamples (in the simple case):

- *Rørdam*

- *Toms* : used the *Cuntz semigroup* to distinguish simple, nuclear C^*-alg. which cannot be distinguished by the conventional Elliott invariant.

The Cuntz semigroup W(.):

- $a, b \in A^+$: $a \precsim b$ if $\exists \{x_n\} \subset A$ such that $a = \lim_{n\to\infty} x_n b x_n^*$. (*Cuntz*)

- $a, b \in M_\infty(A)^+$: $a \precsim b$ if $a \precsim b$ in $M_n(A)$ for some n such that $a, b \in M_n(A)$.

• $a, b \in M_\infty(A)^+$: $a \sim b$ if $a \precsim b$ and $b \precsim a$ (a and b Cuntz equivalent.)

• $W(A)$, the Cuntz semigroup of A, is defined by:

$$W(A) := M_\infty(A)^+/\sim = \{\langle a \rangle : a \in M_\infty(A)^+\}$$

• $W(A)$ = a positively ordered abelian semigroup when equipped with the relations:

$$\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle,\ \langle a \rangle \leq \langle b \rangle \Leftrightarrow a \precsim b,\ a, b \in M_\infty(A)^+ .$$

(*Coward-Elliott-Ivanescu*, Crelle's Journal 2008):

$Cu(A) \cong W(A \otimes \mathcal{K})$:

• closed under suprema of increasing sequences

$Cu(.)$:

• sequentially continuous

Conjecture (*Toms-Winter*, 2007):

Let A = C^*-alg. + unital + sep. + *simple* + non-elementary + nuclear . T.F.A.E.:

1. *A = finite nuclear dimension;*

2. *A = $\mathcal{Z}$-stable (i.e., $A \cong A \otimes \mathcal{Z}$);*

3. *A = strict comparison of positive elements (i.e., whenever $a, b \in A^+$ satisfy $d_\tau(a) < d_\tau(b)$, $\forall \tau \in T(A)$, then $a \precsim b$).*

Important:

Extend "comparison" to the *non-simple case* (e.g., to the *ideal property* case) and prove appropriate "comparison" results.

Definition (*Kirchberg-Rørdam*):

A C^*-alg. A is said to be *purely infinite* if:

(1) A has no characters (or, equivalently, no non-zero abelian quotients), and

(2) $\forall a, b \in A^+$ such that $a \in \overline{AbA} \Rightarrow \exists \{x_n\} \subset A$ such that $a = \lim_{n\to\infty} x_n^* b x_n$ (i.e., $a \precsim b$).

Remark:

The study of purely infinite C^*-alg. was motivated by *Kirchberg*'s classification of the sep., nuclear C^*-alg. that tensorially absorb the Cuntz algebra $\mathcal{O}_\infty$ up to stable isomorphism by an ideal related KK-theory.

Definition:

A C^*-alg. A is said to be an *AH algebra*, if A is the inductive limit C^*-alg. of:

$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \xrightarrow{\phi_{3,4}} \cdots \xrightarrow{\phi_{n-1,n}} A_n \xrightarrow{\phi_{n,n+1}} \cdots$$

with $A_n = \bigoplus_{i=1}^{t_n} P_{n,i} M_{[n,i]}(C(X_{n,i}))P_{n,i}$, where the local spectra $X_{n,i}$ = finite, connected CW complexes, $t_n, [n,i] \in \mathbb{N}$ and each $P_{n,i} \in \mathcal{P}(M_{[n,i]}(C(X_{n,i})))$.

The ideal property

Definition:

A C^*-alg. A is said to have the *ideal property (i.p.)* if each (closed, two-sided) ideal of A is generated (as an ideal) by its projections.

Some remarks and results:

- $A =$ simple + unital $\Rightarrow A =$ i.p.

- RR$(A) = 0 \Rightarrow A =$ i.p.

- (*Sierakowski*): Let (A, G, α) be a C^*-dynamical system, where $G =$ discrete amenable group and the action of G on $\widehat{A}$ is free. Then $A =$ i.p. $\Rightarrow C^*(G, A, \alpha) =$ i.p.

- (*P.-Phillips*): Let $\alpha\colon G \to Aut(A)$ be an action of a finite group on A with the Rokhlin property. Then $A =$ i.p. $\Rightarrow C^*(G, A, \alpha) =$ i.p.

- (*Cuntz-Echterhoff-Li*): If R is a ring of integers in a number field $\Rightarrow$ the semigroup C^*-alg. $C_r^*(R \rtimes R^\times) =$ i.p. (+ purely infinite + RR$(C_r^*(R \rtimes R^\times)) \neq 0$)

• *(K. Stevens)*: Classification of a certain class of AI alg. + i.p.

• *(P.)*: Classification of the AH alg. + i.p. + s.d.g., up to a shape equivalence.

• *(P.)*: Several characterizations of the i.p. for an arbitrary AH alg.

• *(P.)*: If $A = AH$ alg. + i.p. + s.d.g. Then:

(1) sr$(A) = 1$;

(2) $K_0(A) =$ Riesz group + weakly unperforated (in the sense of Elliott).

• *(Gong-Jiang-Li-P.)*: If $A = AH$ alg. + i.p. + no dim. growth. $\Rightarrow$ A can be rewritten as an AH alg. with (special) local spectra of dim ≤ 3.

• (*P.-Rørdam*, J.F.A. 2000): i.p. $\otimes$ i.p. $\neq$ i.p. (even in the sep. case). If at least one of the "factors" is exact, then we have "equality".

• (*P.-Rørdam*, Crelle's Journal 2007): Let $A = C^*$-alg. + sep. + purely infinite. T.F.A.E.:

(1) $A =$ i.p.;

(2) Prim(A) = a basis of compact-open sets.

• (*P.-Rørdam*, Crelle's Journal 2007): Let $A = C^*$-alg. + sep. T.F.A.E.:

(1) $A \otimes \mathcal{O}_2 =$ i.p.;

(2) RR($A \otimes \mathcal{O}_2$) = 0;

(3) Prim(A) = a basis of compact-open sets.

A RIESZ TYPE INTERPOLATION PROPERTY FOR W(.) AND THE IDEAL PROPERTY

Definition (*P.-Perera*):

Let $A = C^*$-alg. We say that the Cuntz semigroup $W(A)$ has the *weak Riesz interpolation by projections property* if:

$\forall a_i, b_i \in M_\infty(A)^+$ such that $\langle a_i \rangle \leq \langle b_j \rangle$ (in $W(A)$), $1 \leq i, j \leq 2$ and $\forall \varepsilon > 0, \exists p \in \mathcal{P}(M_\infty(A))$ and $m \in \mathbb{N}$ such that we have (in $W(A)$):

$$\langle (a_i - \varepsilon)_+ \rangle \leq \langle p \rangle \leq m \langle b_j \rangle, 1 \leq i, j \leq 2.$$

Theorem (*P.-Perera*):

Let $A = C^*$-alg. + i.p. Then, $W(A)$ = weak Riesz interpolation by projections property.

Lemma (*P.-Perera*):

Let $A = C^*$-alg., let I be an ideal of A that is generated (as an ideal) by $\mathcal{P}(I)$ and let $a \in A^+$.

(i) If $a \in I$, then $\forall \varepsilon > 0, \exists p \in \mathcal{P}(M_\infty(A))$ such that $(a - \varepsilon)_+ \precsim p$, where p = a finite direct sum of projections of I.

(ii) $\forall q \in \mathcal{P}(\overline{AaA}), \exists n \in \mathbb{N}$ such that $q \precsim a \otimes 1_n$.

Proof of the Theorem. Let $a_i, b_i \in M_\infty(A)^+$ such that $\langle a_i \rangle \leq \langle b_j \rangle$, $1 \leq i, j \leq 2$. We may suppose that $a_i, b_i \in A^+, 1 \leq i \leq 2$. Let $\varepsilon > 0$. Note that since $a_i \leq a_1 + a_2$, $1 \leq i \leq 2$, a result of *Rørdam* implies that:

$$\langle a_i \rangle \leq \langle a_1 + a_2 \rangle$$

for $i = 1, 2$. Then, by another result of *Rørdam*, for our $\varepsilon > 0, \exists \delta > 0$ such that:

$$\langle (a_i - \varepsilon)_+ \rangle \leq \langle (c - \delta)_+ \rangle, 1 \leq i \leq 2\,, \tag{1}$$

where $c := a_1 + a_2$. Since $\langle a_i \rangle \leq \langle b_j \rangle, 1 \leq i, j \leq 2$, we have that $c \in \overline{Ab_jA}, 1 \leq j \leq 2$, i.e. $c \in I := \overline{Ab_1A} \cap \overline{Ab_2A}$. Note that since $A=$ i.p. and I = ideal of $A \Rightarrow I$ is generated (as an ideal) by $\mathcal{P}(I)$. Then, by the above Lemma $\Rightarrow$ for our $\delta > 0, \exists p \in \mathcal{P}(M_\infty(A))$ such that p = a finite direct sum of projections of I and $\exists m \in \mathbb{N}$ such that:

$$\langle (c - \delta)_+ \rangle \leq \langle p \rangle \leq m \langle b_j \rangle, 1 \leq j \leq 2 \tag{2}$$

Finally, (1) and (2) imply that:

$$\langle (a_i - \varepsilon)_+ \rangle \leq \langle p \rangle \leq m \langle b_j \rangle, 1 \leq i, j \leq 2\,,$$

which ends the proof.

CHARACTERIZATION OF THE IDEAL PROPERTY IN TERMS OF W(.)

Theorem (*P.-Perera*):

Let $A = C^*$-alg. T.F.A.E.:

(i) $A =$ i.p.;

(ii) $\forall a_i, b_i \in A^+$ such that $\langle a_i \rangle \leq \langle b_j \rangle$, $1 \leq i, j \leq 2$ and $\forall \varepsilon > 0$, $\exists p \in \mathcal{P}(M_\infty(A))$ and $\exists m \in \mathbb{N}$ such that $\langle (a_i - \varepsilon)_+ \rangle \leq \langle p \rangle \leq m \langle b_j \rangle$, $1 \leq i, j \leq 2$ and $p =$ a finite direct sum of projections of A;

(iii) $\forall a \in A^+$ and $\forall \varepsilon > 0, \exists p \in \mathcal{P}(M_\infty(A))$ and $m \in \mathbb{N}$ such that $\langle (a - \varepsilon)_+ \rangle \leq \langle p \rangle \leq m \langle a \rangle$ and $p =$ a finite direct sum of projections of A.

A SPECIAL CASE

Theorem (*P.-Perera*):

Let $A = C^*$-alg. + purely infinite + stable. T.F.A.E.:

(i) $A =$ i.p.;

(ii) $\forall a \in A^+, \exists \ \{p_n\} \subset \mathcal{P}(A)$ such that $\langle a \rangle = \sup_{n\in\mathbb{N}} \langle p_n \rangle$ (in $W(A)$);

(iii) $\forall a \in A^+, \ \exists \{q_n\} \subset \mathcal{P}(A)$ such that $\{\langle q_n \rangle\}$ is increasing in $W(A)$ and $\langle a \rangle = \sup_{n\in\mathbb{N}} \langle q_n \rangle$ (in $W(A)$);

(iv) $\forall a \in A^+$, we have that $\overline{AaA} = \overline{\cup_{n\geq 1} I_n}$, where $\{I_n\}$ is an increasing sequence of ideals of A and each I_n is generated (as an ideal) by a single projection.

Proposition (*P.-Perera*):

Let $A = C^*$-alg. $+$ purely infinite $+$ i.p. and let $a \in A^+ \Rightarrow \exists \{p_n\} \subset \mathcal{P}(M_\infty(A))$ such that $\{\langle p_n \rangle\}$ is an increasing sequence in $W(A)$ and $\langle a \rangle = \sup_{n \in \mathbb{N}} \langle p_n \rangle$ (in $W(A)$).

Remark (*P.-Perera*):

(i) Note that if $A = C^*$-alg. $+$ purely infinite $\Rightarrow W(A)$ $=$ Riesz interpolation property. The same conclusion holds for the semigroup $V(A)$ consisting of the Murray-von Neumann equivalence classes $[p]$ of projections in $M_\infty(A)$.
Indeed, let $a_i, b_i \in M_\infty(A)^+$ be such that $\langle a_i \rangle \leq \langle b_j \rangle, 1 \leq i, j \leq 2$ (in $W(A)$). We may assume that $a_i, b_i \in A^+, 1 \leq i \leq 2$. Then, for all i, j:

$$\langle a_i \rangle \leq \langle a_1 + a_2 \rangle \leq \langle a_1 \rangle + \langle a_2 \rangle \leq 2\langle b_j \rangle \leq \langle b_j \rangle \, .$$

($\forall$ non-zero positive element of a purely infinite C^*-alg. is properly infinite).

(ii) For $A = C^*$-alg., denote by:

$$W_{pi}(A) := \{\langle a \rangle \in W(A) \mid a = 0 \text{ or prop. inf. in } M_\infty(A)\} \, .$$

Then the same argument as in (i) shows that $W_{pi}(A)$ $=$ subsemigroup of $W(A)$ with Riesz interpolation.

With this language, a theorem of *Kirchberg-Rørdam* can be rephrased by saying that:

- $A =$ purely infinite $\Leftrightarrow W(A) = W_{pi}(A)$.

COMPARISON OF POSITIVE ELEMENTS AND THE IDEAL PROPERTY

• A *dimension function* on a C^*-alg. A is an additive order preserving function $d : W(A) \to [0, \infty]$. We can also regard d as a function $M_\infty(A)^+ \to [0, \infty]$ that respects the rules $d(a \oplus b) = d(a) + d(b)$ and $a \precsim b \Rightarrow d(a) \le d(b)$ for all $a, b \in M_\infty(A)^+$.

• Define $DF(A)$:= the set of all dimension functions on a C^*-alg. A.

• A dimension function d on A is said to be *lower semi-continuous* if $d(a) = \sup_{\varepsilon>0} d((a-\varepsilon)_+)$ for all $a \in M_\infty(A)^+$.

• Let A = unital C^*-alg. A *(normalized) quasitrace* on A ia a function $\tau : A \to \mathbb{C}$ satisfying:

(i) $\tau(1) = 1$;

(ii) $0 \le \tau(xx^*) = \tau(x^*x)$, for all $x \in A$;

(iii) $\tau(a + ib) = \tau(a) + i\tau(b)$, for all $a, b \in A_{sa}$;

(iv) τ is linear on abelian sub-C^*-alg. of A;

(v) τ extends to a function from $M_n(A)$ to $\mathbb{C}$ satisfying (i)-(iv).

• Define $QT(A)$:= the set of all (normalized) quasitraces on A. This notion was introduced by *Blackadar-Handelman*.

• Given $\tau \in QT(A)$ one may define a map $d_\tau : M_\infty(A)^+ \to [0, \infty]$ by:

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$$

Note that in fact d_τ *takes only real values*: $d_\tau(M_\infty(A)^+) \subseteq [0, \infty)$.

• *Blackadar* and *Handelman* showed that d_τ = lower semicontinuous dimension function on A. Note that for all $p \in \mathcal{P}(M_\infty(A))$ we have that $d_\tau(p) = \tau(p)$.

Definition A (*P.-Perera*):

A unital C^*-alg. A such that $QT(A) \neq \emptyset$ is said to have *weak strict comparison* if it has the property that $a \precsim b$ whenever $a, b \in M_\infty(A)^+$ satisfy the inequality:

$$d(a) < d(b), \forall d \in E \cup \{f \in DF(A) \setminus E : f(b) = 1\}$$

where $E := \{d_\tau : \tau \in QT(A)\}$.

Definition (*P.-Perera*):

A unital C^*-alg. A such that $QT(A) \neq \emptyset$ is said to have *strict comparison of projections* if $p \precsim q$ whenever $p, q \in \mathcal{P}(M_\infty(A))$ satisfy the inequality:

$$\tau(p) < \tau(q), \forall \tau \in QT(A).$$

Theorem A (*P.-Perera*):

Let $A = C^*$-alg. + unital + i.p. + strict comparison of projections + finitely many extremal quasitraces. Let $a, b \in M_\infty(A)^+$ such that:

$$d_\tau(a) < d_\tau(b), \forall \tau \in QT(A)\,.$$

Then $\forall \varepsilon > 0, \exists m \in \mathbb{N}$ such that:

$$(a-\varepsilon)_+ \precsim b \otimes 1_m.$$

Remark (*Rørdam*):

Let $a, b \in A^+$. T.F.A.E.:

(1) $\forall \varepsilon > 0, \exists m \in \mathbb{N}$ such that $(a-\varepsilon)_+ \precsim b \otimes 1_m$;

(2) $a \in \overline{AbA}$.

Corollary A (*P.-Perera*):

Let A = unital + AH alg. + i.p. + finitely many extremal tracial states. Let $a, b \in M_\infty(A)^+$ such that:

$$d_\tau(a) < d_\tau(b), \forall \tau \in T(A).$$

Then $\forall \varepsilon > 0, \exists m \in \mathbb{N}$ such that:

$$(a-\varepsilon)_+ \precsim b \otimes 1_m.$$

Definition:

A positive ordered abelian semigroup W (in particular, the Cuntz semigroup of a C^*-algebra) is said to be *almost unperforated* if $\forall x, y \in W$ and $\forall m, n \in \mathbb{N}$ with $nx \leq my$ and $n > m \Rightarrow x \leq y$.

Theorem B (*P.-Perera*):

Let $A = C^*$-alg. + unital + i.p. + strict comparison of projections + finitely many extremal quasitraces. Assume that $W(A)$ = almost unperforated. Then A = weak strict comparison.

Theorem C (*P.-Perera*):

Let $A = AH$ alg. + unital + i.p. + finitely many extremal tracial states. Assume that $W(A)$ = almost unperforated. Then A = weak strict comparison.

Theorem D (*P.-Perera*):

Let A = AH alg. + unital + i.p. + finitely many extremal tracial states and let B = unital + simple + infinite dimensional AH alg. + no dimension growth + a unique tracial state. Then $A \otimes B$ = weak strict comparison.

Proof. Observe first that since A, B = i.p. and A (or B) = exact, it follows that $A \otimes B$ = i.p. (use, e.g., a result of *P.-Rørdam*). On the other hand, by a result of *Toms-Winter*, B = $\mathcal{Z}$-stable, that is $B \cong B \otimes \mathcal{Z}$, where $\mathcal{Z}$ is the Jiang-Su algebra. Hence the unital AH alg. with the ideal property $A \otimes B$ is $\mathcal{Z}$-stable, i.e. $A \otimes B \cong (A \otimes B) \otimes \mathcal{Z}$, and then a result of *Rørdam* $\Rightarrow W(A \otimes B)$ = almost unperforated. Note that if $T(B) = \{\sigma\} \Rightarrow T(A \otimes B) = \{\tau \otimes \sigma : \tau \in T(A)\}$ and since A = finitely many extremal tracial states $\Rightarrow A \otimes B$ = finitely many extremal tracial states. Now, the fact that $A \otimes B$ = weak strict comparison follows from the previous Theorem.

Remark (*P.-Perera*):

We may say that a unital C^*-alg. A with $QT(A) \neq \emptyset$ has *almost weak strict comparison* if A satisfies all the conditions in the definition of *weak strict comparison* (Definition A), with the only difference that the condition:

$$(*)\ d(a) < d(b), \forall d \in E$$

is replaced by the new condition:

$$(**)\ \exists \varepsilon_0 > 0 \text{ s.t. } d(a) < d((b-\varepsilon_0)_+), \forall d \in E\,,$$

with E as in Definition A above (of course, we still request that $d(a) < d(b), \forall d \in \{f \in DF(A) \setminus E : f(b) = 1\}$).

In the proof of Theorem A we showed, in particular, that in the case when a unital C^*-alg. $A =$ finitely many extremal quasitraces, then $(*) \Rightarrow (**)$. Therefore, in this case, if $A =$ almost weak strict comparison $\Rightarrow A =$ weak strict comparison. Note that if we drop the condition that the C^*-alg. $A =$ finitely many extremal quasitraces (tracial states), the conclusions of Theorem A and of Corollary A remain true if we replace in their hypotheses condition $(*)$ by condition $(**)$ as above. Also, it is easy to see that, if in Theorems B, C and D we *drop the condition* that $A =$ finitely many extremal quasitraces (tracial states) and the condition that B = unique tracial state (in Theorem D), then they *remain true* if we *replace* in their conclusions "weak strict comparison" by "almost weak strict comparison".